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International Journal of Solids and Structures 39 (2002) 2465–2475

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

An efficient 3D stochastic finite element method

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Received 15 July 2001; received in revised form 20 November 2001

Abstract

Real life structural systems are characterized by their inherent or externally induced uncertainties in the design parameters. This study proposes a stochastic finite element tool efficient to take account of these uncertainties. Here uncertain structural parameter is modeled as homogeneous Gaussian stochastic field and commonly used two-dimensional (2D) local averaging technique is extended and generalized for 3D random field. This is followed by Cholesky decomposition of respective covariance matrix for digital simulation. By expanding uncertain stiffness matrix about its reference value, the Neumann expansion method is introduced blended with direct Monte Carlo simulation. This approach involves decomposition of stiffness matrix only once for the entire simulated structure. Thus substantial saving of CPU time and also the scope of tackling several stochastic fields simultaneously are the basic advantages of the proposed algorithm. Accuracy and efficiency of this method with reference to example problem is also studied here and numerical results validate its superiority over direct simulation method or first-order perturbation approach.

Keywords: Neumann expansion; Local averaging; Cholesky decomposition; Monte Carlo simulation; Perturbation

1. Introduction

Modern numerical techniques like the finite element, finite difference, boundary element methods etc. have become quite elaborate and include sophisticated mathematical modeling for analysis. Conventional analysis procedure utilizes selectively representative (minimum or maximum) values of the structural parameters involved. But real life structural systems employed in civil, mechanical, aerospace and offshore fields are characterized by inherent randomness in the definition of their parameters described temporally and spatially. The major sources of such randomness encountered in physical environment can be identified in uncertainty of material properties, loading conditions, geometric profiles etc. Scatters in characteristics of these system parameters have been noticed even with the best quality control and those can precisely be described only probabilistically. In this context, it is important to emphasize the fact that increased sophistication in the stress analysis must not be trade of by simplifying in modeling the structural system

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0020-7683/02/$ - see front matter
PII: S0020-7683(02)00080-X

parameters. The analysis should be consistent in the sense that mechanical modeling, stress analysis and safety evaluations are carried out at least at the same degree of sophistication. This can not be achieved properly through deterministic analysis.

The numerical tool required for stochastic structural analysis should primarily be concerned with quantifying uncertainties in structural response i.e. displacements, stresses etc. The most developed one i.e. the finite element method provides the necessary modeling flexibility. Standard deterministic form of the finite element tool modified to the stochastic finite element method (SFEM) has been developed to analyze system stochasticity problems. Extensive reviews regarding this are found in Vanmarcke et al. (1986), Benaroya and Rehak (1988), and Schuëller (1997). In stochastic finite element method three basic techniques are generally practiced. The most widely used one is the perturbation technique (Hart and Collins, 1970; Handa and Karrholm, 1975; Hisada and Nakagiri, 1980; Nakagiri and Hisada, 1980, 1983; Liu et al., 1986). It is quite a simple method utilizing first-order Taylor series expansion to formulate linear relationship between statistical characteristics of the response and input random structural parameters. But involvement in the computation time for terms higher than first-order is prohibitively high and not practical. This restricts the applicability of the method to the problems associated with small variability. The Monte Carlo simulation (Shinozuka and Astill, 1972; Shinozuka, 1987; Yamazaki and Shinozuka, 1989) is another method, which is frequently used in system stochasticity analysis. The major advantage of simulation method is that an accurate solution can be obtained and higher order statistical moments and probability distribution of the response quantities can be computed. But it involves abnormally high amount of CPU time. The third approach is known as the Neumann Expansion method, which has been rarely used in the field of structural mechanics (Shinozuka and Deodatis, 1988). But the method is found to be effective when coupled with the Monte Carlo simulation technique (Yamazaki et al., 1988; Chakraborty, 1997).

The present study primarily focus on the SFEM in determining response variability for three-dimensional (3D) static problems subjected to spatial uncertainties of the material properties. To accomplish this, the random fields are discretised by local averaging technique and then simulated by Cholesky decomposition of the respective covariance matrix. Stochastic modeling employing local averaging for 2D case is readily available in the literature (Vanmarcke, 1977; Vanmarcke et al., 1986; Vanmarcke and Grigoriu, 1983). But it appears that no work is reported as yet for 3D case. Following the basic principle (Vanmarcke, 1977; Vanmarcke, 1983; Vanmarcke et al., 1986; Zhu et al., 1992) local averaging model is extended for three dimension. The finite element solution has been obtained for response variability utilizing the Neumann expansion technique within the framework of Monte Carlo simulation and analysis is also carried out for comparison in terms of accuracy and efficiency.

2. Stochastic modeling by local averaging method

Several methods of discretization using continuous and discontinuous representation of stochastic field are reported in literature (Vanmarcke, 1977; Yamazaki et al., 1988; Ghanem and Spanos, 1991). Continuous representation involves solution of the integral eigenvalue problem, which may not have a closed form solution for realistic covariance function. But in discontinuous representation, the stochastic finite element models based on local averaging technique found to yield more rapid convergence than the mid point method (Zhu et al., 1992). Moreover, detailed knowledge about the correlation function of the random field is not essential. If it is available, the use of direct variance reduction function can be obtained. In the present section, 2-D local averaging technique is extended for 3-D averaging.

In local averaging technique the field variable over an element is approximated by spatial average. A homogeneous Gaussian random scalar field $\alpha(x, y, z)$ defined over the domain Ω is assumed, characterized by its mean $\bar{\alpha}(x, y, z)$, variance σ^2 and variance function $\rho(r_x, r_y, r_z)$ where $r_x = x - \bar{x}$, $r_y = y - \bar{y}$ and

$r_z = z - \bar{z}$. The local averages of the field over a volume V_i centered at (x_i, y_i, z_i) having sides L_{xi}, L_{yi} and L_{zi} parallel to 'x', 'y' and 'z' axis respectively, is defined as:

$$\alpha(x_i, y_i, z_i) = \frac{1}{L_{x_i} L_{y_i} L_{z_i}} \int_{x_i-(L_{x_i}/2)}^{x_i+(L_{x_i}/2)} \int_{y_i-(L_{y_i}/2)}^{y_i+(L_{y_i}/2)} \int_{z_i-(L_{zi}/2)}^{z_i+(L_{zi}/2)} \alpha(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \tag{1}$$

If the correlation of the field is assumed to be quadrant symmetric, the mean vector, variance and the covariance of the spatial averages between any two volume elements V_i and V_j can be obtained as:

$$E(\alpha) = \bar{\alpha} \tag{2}$$

$$\mathrm{var}(\alpha) = \sigma \gamma(L_x, L_y, L_z) \tag{3}$$

$$\mathrm{cov}(\alpha_i, \alpha_j) = \sigma \frac{1}{8 V_i V_j} \sum_{k,l,m=0}^{3} (-1)^{k+l+m} (L_{xk} L_{yl} L_{zm})^2 \gamma(L_{xk}, L_{yl}, L_{zm}) \sigma \tag{4}$$

where $\sigma = \mathrm{diag}[\sigma_1, \sigma_2, \ldots\ldots \sigma_N]$, σ_i is the standard deviation of the ith random parameter and

$$\gamma(L_{xk}, L_{yl}, L_{zm}) = \frac{8}{L_{xk} L_{yl} L_{zm}} \int_0^{L_{xk}} \int_0^{L_{yl}} \int_0^{L_{zm}} \left(1 - \frac{r_x}{L_{xk}}\right)\left(1 - \frac{r_y}{L_{yl}}\right)\left(1 - \frac{r_z}{L_{zm}}\right) \rho(r_x, r_y, r_z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \tag{5}$$

Here L_{xk}, L_{yl} and L_{zm} $(k, l, m = 0, 1, 2, 3)$ are the distances characterizing relative positions of any two discretised volumes V_i and V_j as shown in Fig. 1 And $\gamma(L_{xk}, L_{yl}, L_{zm})$ is normalized variance function of the local averages $\alpha(x, y, z)$ over the volume with sides L_{xk}, L_{yl} and L_{zm}. This function depicts the dependence of the variance of spatial averages on the size of averaging element. The 1D variance function corresponding to $\rho(r) = \exp\left[-(r/b)^2\right]$ is

$$\gamma(r) = \left(\frac{b}{r}\right)^2 \left[\frac{r}{b}\sqrt{\pi}\phi\left(\frac{b}{r}\right) + \mathrm{e}^{-\left(\frac{r}{b}\right)^2} - 1\right] \tag{6}$$

where $\Phi(\cdot)$ is the error function. Its value increases from zero to one as the argument increases from zero to infinity and 'b' is the correlation parameter. If $\alpha(x, y, z)$ is separable, i.e., $\rho(r_x, r_y, r_z) = \rho(r_x)\rho(r_y)\rho(r_z)$, 1D expression for variance reduction function can well be modified for 3D purpose and becomes simply the product of three 1D factors, $\gamma(L_{xk}, L_{yl}, L_{zm}) = \gamma(L_{xk})\gamma(L_{yl})\gamma(L_{zm})$. This 1D result for computing the variance

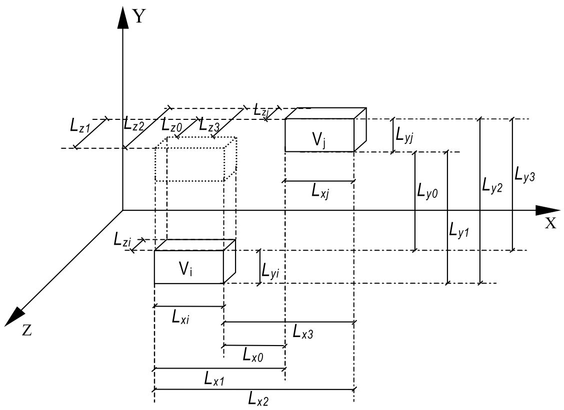

Fig. 1. Definition of distances characterizing relative positions of any two cubic volumes V_i and V_j.

reduction function can be easily used for the exponential covariance function of the following form, which has been taken for present numerical study.

$$\rho(r_x, r_y, r_z) = \exp\left[-\left\{\left(\frac{r_x}{b_x}\right)^2 + \left(\frac{r_y}{b_y}\right)^2 + \left(\frac{r_z}{b_z}\right)^2\right\}\right] \tag{7}$$

For a particular analytic expressions corresponding normalized variance function can be computed. Various form of analytic expressions i.e. triangular, exponential etc. characterized by a parameter (correlation parameter) have been reported (Vanmarcke, 1977). However suitability of any model can be justified by fitting actual experimental data. As no experimental data is available to ascertain the relative merits of alternative models, exponential models have been selected with the purpose of illustrating the analytical procedure. Note that if the field is not homogeneous and or quadrant symmetric or the local averaging domain is non rectangular, then the Gaussian quadrature can be used (Zhu et al., 1992) to compute the covariance matrix of the random vector field.

3. Simulation of stochastic field

If there are N finite elements in the structure, correlated zero mean random vector $\{\alpha\} = \{\alpha_1, \alpha_2, \ldots\ldots \alpha_N\}^{\mathrm{T}}$ of the random field $\alpha(x, y, z)$ can be obtained as:

$$\{\alpha\} = [L]\{Z\} \tag{8}$$

in which $\{Z\} = \{Z_1, Z_2, \ldots\ldots Z_N\}^{\mathrm{T}}$ is a vector comprising N independent Gaussian random variates with zero mean and unit standard deviation and $[L]$ is the lower triangular matrix derived through Cholesky decomposition of $\mathrm{cov}(\alpha_i, \alpha_j)$. Once the Cholesky decomposition is over, the sample vector of any random design variable can easily be obtained for any number of simulations desired (Yamazaki et al., 1988). In general, the sample size is guided by the desired degree of accuracy of the mean and standard deviation. However a fairly large size of the ensemble is required for the simulated covariance matrix approach the target covariance matrix, satisfying the original covariance property (Vanmarcke et al., 1986).

4. General FEM formulation

The standard displacement based finite element discretised version (Cook et al., 1988) of the equilibrium equation for linear static problem can be written as:

$$[K(h)]\{u(h)\} = \{F(h)\} \tag{9}$$

In which the load vector $\{F(h)\}$ and the overall structural stiffness matrix $[K(h)]$, both are function of design parameter h, defined as:

$$\begin{aligned} [K(h)] &= \sum_{e=1}^{N} [k(h)]_e \\ \{F(h)\} &= \sum_{e=1}^{N} \{f(h)\}_e \end{aligned} \tag{10}$$

where summation symbols indicate that contributions from all the elements of the structure need to be included. The element stiffness matrix $[k(h)]_e$ and the element load vector $\{f(h)\}_e$ in above equations are defined as:

$$
\begin{aligned}
[k(h)]_e &= \int_v [B]^{\mathrm{T}} [D]_e [B]\, \mathrm{d}v \\
\{f(h)t\}_e &= \int_s [N]^{\mathrm{T}} \{\phi\}\, \mathrm{d}s
\end{aligned}
\tag{11}
$$

After applying proper boundary conditions, the unknown displacements $\{u(h)\}$ are obtained by solving the equilibrium equation.

5. Stochastic finite element formulation

In Eq. (11), the elasticity matrix $[D]_e$ is random in nature due to the uncertainty in the parameter h and can be written in the following form:

$$
[D]_e = h_\alpha [C]_e \tag{12}
$$

where h_α is a random field representing the modulus of elasticity over the domain Ω_e, $[C]_e$ is a deterministic matrix and $\boldsymbol{\Omega}_e \in \boldsymbol{\Omega}$. As $[D]_e$ is random, the stiffness matrix $[K]_e$ in equilibrium equation (9) is also random in nature. Here the design variable is decomposed into its mean and fluctuating component as, $h = h_0 + \Delta h$, where Δh is obtained at elemental through digital simulation of the stochastic field. The random stiffness matrix can be spilt up into mean and deviatoric parts as:

$$
\begin{aligned}
[k(h)]_e &= [k(h_0)]_e + [k(\Delta h)]_e \\
[K(h)] &= [K(h_0)] + [K(\Delta h)]
\end{aligned}
\tag{13}
$$

where

$$
[k(h)]_e = C_\alpha \int_\Omega [B]^{\mathrm{T}} [C]_e [B]\, \mathrm{d}\Omega \quad \text{and} \quad [k(\Delta h)]_e = \alpha_e [k(h_0)]_e \tag{14}
$$

Here $[B]$ is the standard strain displacement matrix. Though $[B]$ has been taken in present analysis deterministic, it is more elegant to introduce the concept of stochastic shape function (Dasgupta, 1995) in the derivation of strain displacement, particularly for larger stochastic variation. Once the mean and deviatoric part of element stiffness matrices is generated, these can be assembled in usual procedure as done in deterministic finite element method to obtain the global stiffness matrices.

6. Direct Monte Carlo solution

Implementation of the Monte Carlo simulation method consists of numerically simulating a population corresponding to the random quantities in the physical system, solving the deterministic problem associated with each sample of that population and obtaining a population corresponding to the random response quantities. The random vector $\{\alpha\} = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)^{\mathrm{T}}$, which represents the variation of material property or load etc. as may be the case, can be generated using Eq. (8) Subsequently, the structural stiffness matrix or load vector is generated. Thus a sample global stiffness matrix and load vector are formed on the basis of the sample stochastic fields. The response of the structure is then determined by relying on the standard finite element algorithm. This procedure is repeated several times to produce an ensemble of structural response. The statistical algorithms are utilized to compute various useful statistical moments of the response quantities from the ensemble responses.

7. Neumann expansion solution

In direct Monte Carlo simulation method, the random stiffness matrix needs to be inverted for each simulated structure leading to an enormous amount of CPU time. The Neumann expansion technique has been adopted to avoid the repeated inversion of the random stiffness matrix. Here the inversion of random stiffness matrix $[K(h)]$ takes the following form:

$$\begin{aligned}[K(h)]^{-1} &= ([K(h_0)] + [K(\Delta h)])^{-1} = ([I] + [P])^{-1}[K(h_0)]^{-1} \\ &= ([I] - [P] + [P]^2 - [P]^3 + \cdots)[K(h_0)]^{-1} = \left(\sum_{n=0}^{\infty}[-P]^n\right)[K(h_0)]^{-1} \end{aligned} \tag{15}$$

where $[P] = [K(h_0)]^{-1}[K(\Delta h)]$. Substitution of Eq. (11) into Eq. (9) yields the response vector as,

$$\begin{aligned}\{u(h)\} &= \left(\sum_{n=0}^{\infty}[-P]^n\right)[K(h_0)]^{-1}\{F(h)\} = \Big([I] - [P] + [P]^2 - [P]^3 + \cdots\Big)\{u_0\} \\ &= \{u_0\} - \{u_1\} + \{u_2\} - \{u_3\} + \cdots \end{aligned} \tag{16}$$

where $\{u_0\} = [K(h_0)]^{-1}\{F(h)\}$. The above series solution is equivalent to the solution to following recursive equation:

$$[K(h_0)]\{u_i^r\} = [K(\Delta h)]\{u_{i-1}^r\} \tag{17}$$

Once the inversion of deterministic part of the stiffness matrix is computed, $\{u(h)\}$ can be computed for each simulated load vector. Then Eq. (17) can be used iteratively to obtain the random displacement for each simulated sample structure without further decomposition of the stiffness matrix. The expansion series in Eq. (16) may be terminated after few terms depending upon the convergence and accuracy requirement of the solution. The most outstanding feature of the method is that single matrix (deterministic part only) factorisation is required for all the sample structures. Therefore, the computational time can be reduced considerably.

The Neumann expansion used in Eq. (15) will converge if the absolute values of all the eigenvalues of $[P]$ matrix are less than 1.0. For the present study Gaussian model has been selected for stochastic representation of modulus of elasticity, though non-Gaussian field models are also existent (Sobczyk et al., 1996). The Gaussian models have limitations where design parameters experience large variations. Again assumption of Gaussian distribution implies the possibility of generating negative values of elastic properties. As truncated Gaussian distribution has been used for generation of random sample functions of the uncertain parameter, these difficulties can be circumvented and the convergence criterion is automatically satisfied. However this convergence criterion can easily be met irrespective of largeness of each component of the deviatoric part of matrix $[K(h)]$ (Yamazaki et al., 1988).

8. Perturbation method

In perturbation method, all the vectors and matrices in the system equation (9) are expanded employing Taylor's series as:

$$\left([K(h_0)] + \sum_{i=1}^{N}\frac{\partial}{\partial h}[K_i(h_0)]\Delta h\right)\left(\{u(h_0)\} + \sum_{i=1}^{N}\frac{\partial}{\partial h}\{u_i(h_0)\}\Delta h\right) = \left(\{F(h_0)\} + \sum_{i=1}^{N}\frac{\partial}{\partial h}\{F_i(h_0)\}\Delta h\right) \tag{18}$$

Neglecting higher order terms and equating terms of same order, following set of recursive equations are obtained

$$\begin{aligned}[K(h_0)]\{u(h_0)\} &= \{F(h_0)\}\\ [K(h_0)]\{u_i^I(h_0)\} &= -\frac{\partial}{\partial h}[K_i(h)]\{u(h_0)\}\end{aligned} \tag{19}$$

where $\{u_i^I(h_0)\} = \partial/\partial h\{u_i(h_0)\}$ for ith element. The expected values and covariance of $\{u(h)\}$ are estimated as

$$E\{u(h)\} = \{u(h_0)\} \tag{20}$$

$$\mathrm{Cov}[\{u(h)\},\{u(h)\}] = E[(\{u(h)\} - \{u(h_0)\})(\{u(h)\} - \{u(h_0)\}^{\mathrm{T}})] = \sum_{i=1}^{N}\sum_{j=1}^{N}\{u_i^I(h_0)\}\{u_j^I(h_0)\}^{\mathrm{T}}E[h_i,h_j] \tag{21}$$

The variance vector is the diagonal component of $\mathrm{Cov}[\{u(h)\},\{u(h)\}]$ and $E[h_i,h_j]$ is determined from the auto-correlation function of the underlying stochastic field of h.

9. Numerical example

To elucidate the stochastic finite element formulation with respect to the accuracy and efficiency, an L-shaped beam problem subjected to concentrated tip loading of 80 kN is taken. The 3D finite element model is shown in Fig. 2. Spatially varying modulus of elasticity is considered. Mean value of Young's modulus (E) is taken as 200 GPa and deterministic Poisson's ratio 0.30. Here the Young's modulus is modeled as 3D, stationary, zero-mean Gaussian process:

$$E = E_0(1+\Delta E), \quad \text{where } \Delta E = E_0 \cdot \alpha^E \tag{22}$$

which is sufficiently characterized by its mean E_0, and the random deviatoric part α_k^E is obtained through digital simulation as described in Eq. (8). The correlation parameter in all the three directions is taken as half (1/2) the element length. The results of the numerical examples are provided at the locations where large outputs (tip deflections) are expected. The results of the direct Monte Carlo simulation and first-order perturbation have been shown in the same figure and table for the ease of comparison.

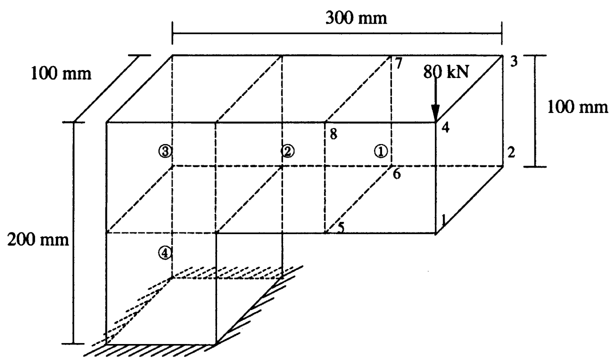

Fig. 2. Finite element discretization of the beam.

10. Results and conclusions

The size of the ensemble should be large enough to obtain statistically stable result. As it is observed from Fig. 3 that fluctuation of standard deviation of deflection is negligibly small after 200-simulation, number of simulation is fixed at 200. The comparison of convergence of expectation as well as standard deviation of displacement with increasing orders in Neumann expansion is shown in Figs. 4 and 5. It reveals that rate of convergence is substantially improved when the order of expansion is increased from one to two. Beyond third order expansion, results approach close to those obtained from direct simulation but

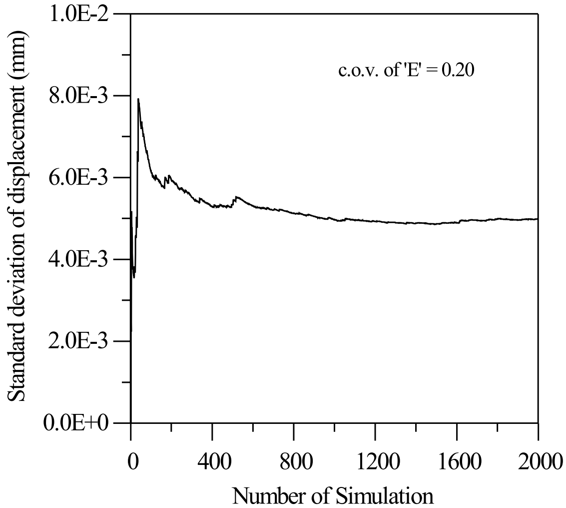

Fig. 3. Fluctuation of standard deviation of displacement.

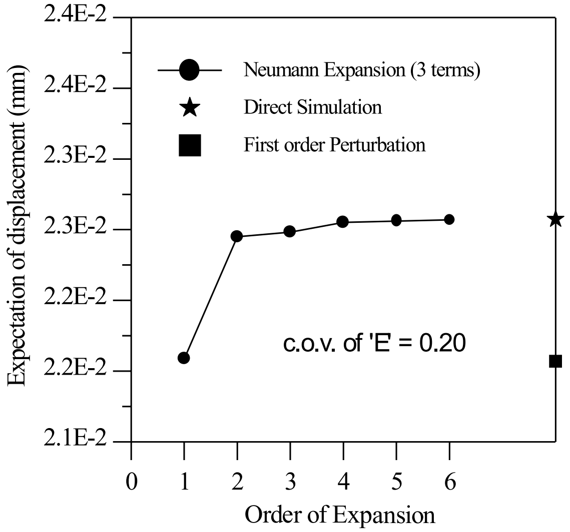

Fig. 4. Convergence of expectation of displacement.

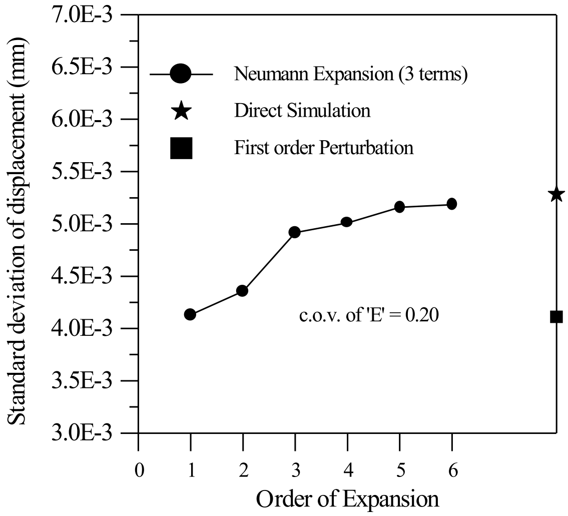

Fig. 5. Convergence of standard deviation of displacement.

with a slow convergence rate. The rate of convergence of the standard deviation is seen to be slower than expectation value. This is partly due to the fact that the standard deviation results only from deviatoric component of the sample whereas the expectation consists of mean value and deviatoric components (Shinozuka and Yamazaki, 1988). However the rate of convergence of response may differ from sample to sample.

The expectation and standard deviation of displacement obtained from direct simulation and Neumann expansion are compared with perturbation results for varying c.o.v. as input parameter as shown in Figs. 6 and 7. The comparison shows that for small range of c.o.v. third order Neumann expansion is sufficient for convergence of expectation as well as standard deviation of displacement. But for convergence of standard

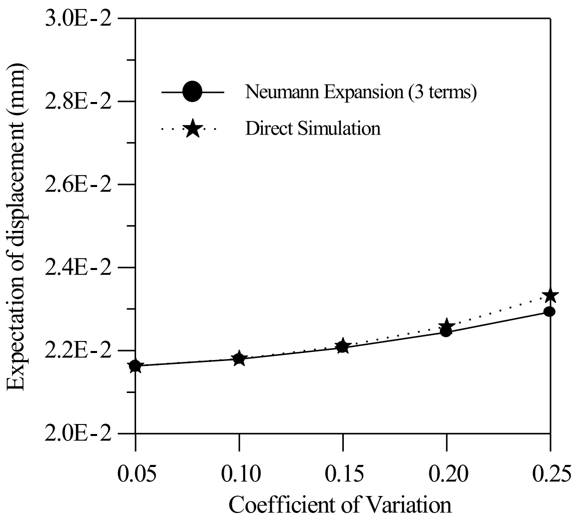

Fig. 6. Comparison of expectation of displacement.

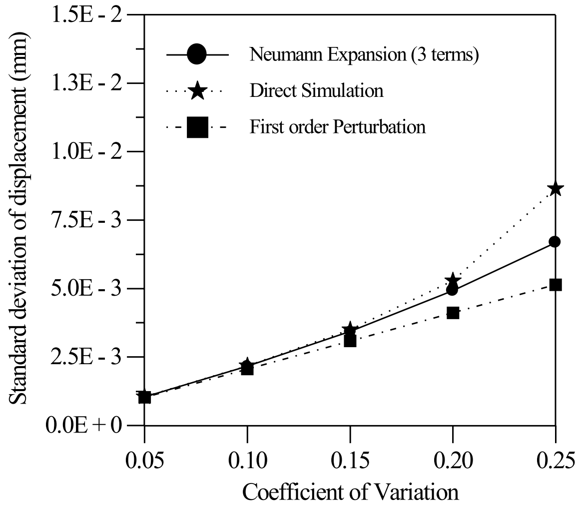

Fig. 7. Comparison of standard deviation of displacement.

deviation at higher c.o.v., higher order Neumann expansion is required. It is also noted that the first order perturbation result is close to that of the first order Neumann expansion. This implies that the accuracy of the perturbation method is comparable to the Neumann expansion simulation method of the same order. Thus first order perturbation method is effective if the c.o.v. of input random parameter is small. Incorporation of even the second order term in perturbation complicates the computation procedure to such an extent that the computation time becomes prohibitively high and even more than the direct simulation (Vanmarcke et al., 1986; Shinozuka and Yamazaki, 1988). Normalized computation time is compared in Fig. 8. For a particular c.o.v., computation time increases with the increase in the order of Neumann Expansion. After five terms, this method takes more CPU time than the direct simulation and is found to be ineffective in terms of computation time. However, time saving depends on the order of expansion, numbers

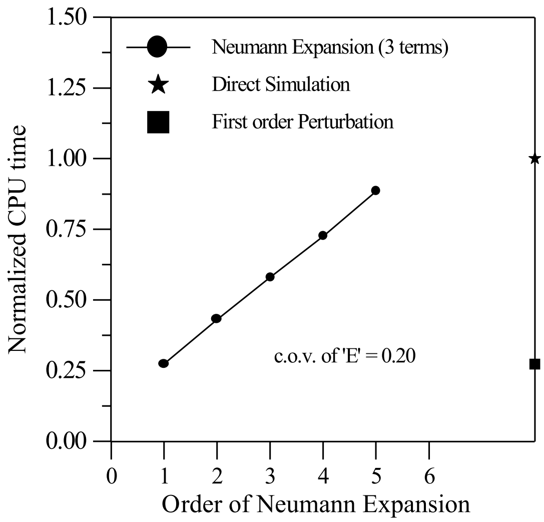

Fig. 8. Comparison of normalized CPU time.

of degrees of freedom in the system, variability of the input and required degrees of accuracy. It has been observed, if the order of Neumann expansion is fixed and not too large, the method will be much more advantageous in terms of CPU time, as the number of degrees of freedom increases.

References

Benaroya, H., Rehak, M., 1988. Finite element methods in probabilistic structural analysis: a selective review. Applied Mechanics Review 41 (5), 201–213.

Chakraborty, S., 1997. Stochastic finite element simulation of structures and reliability assessment. Ph.D. Thesis, Indian Institute of Technology, Kharagpur, India.

Cook, R.D., Malkus, D.S., Plesha, M.E., 1988. Concepts and Application of Finite Element Analysis, third ed. Wiley, New York.

Dasgupta, G., 1995. Stochastic Green's functions. In: Spanos, P.D. (Ed.), Computational Stochastic Mechanics. Balkema, Rotterdam, pp. 511–520.

Ghanem, R.G., Spanos, P.D., 1991. Stochastic Finite Element Analysis: A Spectral Approach. Springer-Verlag, New York.

Handa, K., Karrholm, G., 1975. Application of finite element methods in the statistical analysis of structures. CHALMERS TEKNISKA HOGSKOLA Avd. for byggnadskonstruktion.

Hart, G.C., Collins, J.D., 1970. The treatment of randomness in finite element modelling. SAE, Inc., National Aeronautic and Space Engineering and Manufacturing Meeting, Los Angeles, CA, pp. 1–12.

Hisada, T., Nakagiri, S., 1980. A note on stochastic finite element method (part 2)–variation of stress and strain caused by fluctuations of material properties and geometrical boundary conditions. SEISAN-KENKYU 32 (5), 28–31.

Liu, W.K., Belytschko, T., Mani, A., 1986. Random field finite elements. International Journal for Numerical Methods in Engineering 23, 1831–1845.

Nakagiri, S., Hisada, T., 1980. A note on stochastic finite element method (part 1)–variation of stress and strain caused by shape fluctuation. SEISAN-KENKYU 32 (2), 39–42.

Nakagiri, S., Hisada, T., 1983. A note on stochastic finite element method (part 6)–an application in problems of uncertain elastic foundation. SEISAN-KENKYU 35 (1), 20–23.

Schuëller, G.I., 1997. A state-of-the-art report on computational stochastic mechanics. Probabilistic Engineering Mechanics 12 (4), 197–321.

Shinozuka, M., Astill, J., 1972. Random eigenvalue problems in structural analysis. AIAA Journal 10 (4), 37–40.

Shinozuka, M., 1987. In: Schuëller, G.I., Shinozuka, M. (Eds.), Stochastic Fields and their Digital Simulation–Stochastic Methods in Structural Dynamics, Dordrechts, Netherlands, pp. 93–133.

Shinozuka, M., Deodatis, G., 1988. Response variability of stochastic finite element systems. Journal of Engineering Mechanics, ASCE 114 (3), 237–265.

Shinozuka, M., Yamazaki, F., 1988. Stochastic finite element analysis an introduction. In: Ariaratnam, S.T., Schuëller, G.I., Elishakoff, I. (Eds.), Stochastic Structural Dynamics. Elsevier Science, Amsterdam, pp. 241–291.

Sobczyk, K., Wedrychowicz, S., Spencer, B.F., 1996. Dynamics of structural systems with spatial randomness. International Journal of Structures 33 (11), 1651–1669.

Vanmarcke, E.H., 1977. Probabilistic modeling of soil profiles. Journal of Geotechnical Division, Proceedings of the ASCE 103 (GT11), 1227–1246.

Vanmarcke, E.H., Grigoriu, M., 1983. Stochastic finite element analysis of simple beams. Journal of Engineering Mechanics, ASCE 109 (5), 1203–1214.

Vanmarcke, E.H., 1983. Random Fields: Analysis and Synthesis. MIT Press, Cambridge, MA.

Vanmarcke, E., Shinozuka, M., Nakagiri, S., Schuëller, G.I., Grigoriu, M., 1986. Random fields and stochastic finite elements. Structural Safety 3 (3), 143–166.

Yamazaki, F., Shinozuka, M., Dasgupta, G., 1988. Neumann expansion for stochastic finite element analysis. Journal of Engineering Mechanics, ASCE 114 (8), 1335–1354.

Yamazaki, F., Shinozuka, M., 1989. Statistical preconditioning in simulation of stochastic vector. Fifth International Conference on Structural Safety and Reliability, ICOSSAR, pp. 129–136.

Zhu, W.Q., Ren, Y.J., Wu, W.Q., 1992. Stochastic FEM based on local averages of random vector fields. Journal of Engineering Mechanics, ASCE 118 (3), 496–511.